MINIMAL GENERATING SETS FOR THE FIRST SYZYGIES OF A MONOMIAL IDEAL

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The object of this note is to produce two minimal generating sets of the first syzygies of a monomial ideal, given the minimal generating set of the ideal. In the first section we set up notation and give a preliminary result. In the second section we desribe one small generating set that is not quite minimal. In the third section we describe a different but analogous small generating set. In the final section we describe how to reduce both of these to minimal generating sets.

1. Notation

Let $R = k[x_1, x_2, \dots, x_n]$ be the polynomial ring in n variables over a field k. Let I be a monomial ideal in R. Let $G(I)$ be the minimal generating set of I. For any monomial m in the *lcm-lattice* of $G(I)$, (i.e. the least common multiple lattice of $G(I)$ ordered by divisibility), define

$$\Gamma_m = \{\tau : \tau \text{ is a square free monomial, } \tau | m, \text{ and } \tfrac{m}{\tau} \text{ is in } I\}$$

where

$$\frac{m}{\tau}$$

means the monomial obtained by dividing m by τ.

Also define

$$L_{<m} = \{S : S \subset G(I), n_S | m, \text{ but } n_S \neq m\}$$

where n_S is the least common multiple of the monomials in S. Clearly, n_S is in the *lcm-lattice* of $G(I)$. The vertices of $L_{<m}$ may be identified with the elements of the minimal generating set, $G(I)$.

Γ_m is an abstract finite simplicial complex. (Think of the square free monomials as sets, where divisibility is replaced by containment.) The facets (maximal faces) of Γ_m are among the faces of the form

$$\sqrt{\frac{m}{\gamma}}$$

where γ is in $G(I)$ and $\gamma | m$. Here the root sign

$$\sqrt{n}$$

stands for the largest square free monomial dividing the monomial n. This is sometimes called the *support* of n.

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$L_{<m}$ is also an abstract finite simplicial complex. (Use the usual containment relation.)

For each Γ_m which is disconnected, let $C_{1,m}, C_{2.m}, \ldots, C_{k_m,m}$ be the connected components (as finite simplicial complexes).

Select in $C_{i,m}$, for each i, one facet of the form $\sqrt{\frac{m}{\gamma}}$. From all the $\gamma' s$ such that $\sqrt{\frac{m}{\gamma}}$ equals this face select one and label it $\gamma_{i,m}$. Clearly, we can make this choice because all facets are of this form.

(Later we will make similar, but somewhat more natural choices for $L_{<m}$.)

For each disconnected Γ_m let

$$Y_m = \left\{ \frac{m}{\gamma_{i,m}} \otimes \gamma_{i,m} - \frac{m}{\gamma_{j,m}} \otimes \gamma_{j.m} \right\} \subset R \otimes_k R$$

for all pairs i, j such that $1 \leq i < j \leq k_m$. (If Γ_m is connected let $Y_m = \phi$, the empty set.)

Proposition 1. *For each m such that Γ_m is disconnected and for each pair $\gamma_{i,m}, \gamma_{j,m}$ where $i \neq j$,*

$$m = \operatorname{lcm}(\gamma_{i,m}, \gamma_{j,m})$$

where the right hand side means the least common multiple of the two $\gamma' s$.

Proof. If not, there exists a variable, x, such that $x | \frac{m}{\gamma_{i,m}}$ and $x | \frac{m}{\gamma_{j,m}}$. But then the faces $\sqrt{\frac{m}{\gamma_{i,m}}}$ and $\sqrt{\frac{m}{\gamma_{j,m}}}$ overlap and cannot be in distinct components. $\square$

Now, for all n and m in the *lcm-lattice*, let

$$Y_{\leq m} = \bigcup_{n|m} Y_n$$

$$Y = \bigcup_{n} Y_n$$

For all $\gamma \neq \gamma\prime$ in $G(I)$ let $n_{\gamma,\gamma\prime}$ be the $lcm(\gamma, \gamma\prime)$.

Consider the set

$$\left\{ \frac{n_{\gamma,\gamma\prime}}{\gamma} \otimes \gamma - \frac{n_{\gamma,\gamma\prime}}{\gamma\prime} \otimes \gamma\prime \right\}$$

From Diana Taylor's resolution we conclude that this set generates the first syzygies of the ideal I.

For a given m in the *lcm-lattice* consider all pairs $\gamma, \gamma\prime$ such that $n_{\gamma,\gamma\prime} = m$.

Let

$$T_m = \left\{ \frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma\prime} \otimes \gamma\prime \right\}$$

quantified over all such pairs.

Let

$$T = \bigcup_{m} T_m$$

Clearly T is the above set that generates the first syzygies.

If every element of a subset Q of an R module is a linear combination over R of elements of another subset P we shall say that "P spans Q". We shall also say that elements of Q are "in the span of P".

2. First Generating Set

Theorem 1. *Y spans T.*

Proof. We will prove by induction on the *lcm-lattice* that

$$Y_{\leq m} \text{ spans } T_m$$

Clearly this will suffice to prove the theorem.

Let γ and $\gamma\prime$ be any pair of elements in $G(I)$. Let $m = n_{\gamma,\gamma\prime}$ and consider the element

$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma\prime} \otimes \gamma\prime$$

It suffices to prove that this is in the span of $Y_{\leq m}$.

Case 1:
$\sqrt{\frac{m}{\gamma}}$ and $\sqrt{\frac{m}{\gamma\prime}}$ are faces of the same connected component of Γ_m.

Then there exists a sequence of elements of $G(I)$,

$$\gamma = \gamma_1, \gamma_2, \ldots, \gamma_l = \gamma\prime$$

such that $\sqrt{\frac{m}{\gamma_i}}$ and $\sqrt{\frac{m}{\gamma_{i+1}}}$ overlap for $i = 1, 2, \ldots, l-1$.

So, for each pair γ_i, γ_{i+1} there exists a variable x_i, such that

$$x_i | \frac{m}{\gamma_i} \text{ and } x_i | \frac{m}{\gamma_{i+1}}$$

for $i = 1, 2, \ldots, l-1$. Hence $n_{\gamma_i,\gamma_{i+1}} \neq m$

Let $n_{\gamma_i,\gamma_{i+1}} = n_i$, for $i = 1, 2, \ldots, l-1$. Then $n_i | m$, and $n_i \neq m$. By induction we assume that $Y_{\leq n_i}$ spans T_{n_i}.

Now,

$$\frac{n_i}{\gamma_i} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in T_{n_i} by the definition of T_{n_i}. We have that

$$Y_{\leq n_i} \subset Y_{\leq m}$$

since $n_i | m$.

So,

$$\frac{n_i}{\gamma_i} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in the span of $Y_{\leq m}$ for $i = 1, 2, \ldots, l-1$.

Thus

$$\frac{m}{n_i} \left(\frac{n_i}{\gamma_i} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1} \right)$$

which equals

$$\frac{m}{\gamma_i} \otimes \gamma_i - \frac{m}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in the span of $Y_{\leq m}$.

But

$$\sum_{i=1}^{l-1}\left(\frac{m}{\gamma_i}\otimes\gamma_i-\frac{m}{\gamma_{i+1}}\otimes\gamma_{i+1}\right)$$

telescopes to

$$\frac{m}{\gamma}\otimes\gamma-\frac{m}{\gamma\prime}\otimes\gamma\prime$$

. So this is also in the span of $Y_{\leq m}$. Thus Case 1 is proved.

Case 2:

$\sqrt{\frac{m}{\gamma}}$ and $\sqrt{\frac{m}{\gamma\prime}}$ are faces of different connected components of Γ_m. Say

$$\sqrt{\frac{m}{\gamma}} \text{ is in } C_{j,m}$$

and

$$\sqrt{\frac{m}{\gamma\prime}} \text{ is in } C_{k,m}$$

Thus $\sqrt{\frac{m}{\gamma}}$ and $\sqrt{\frac{m}{\gamma_{j,m}}}$ are in the same connected component $C_{j,m}$. Hence by Case 1,

$$\frac{m}{\gamma}\otimes\gamma-\frac{m}{\gamma_{j,m}}\otimes\gamma_{j,m}$$

is in the span of $Y_{\leq m}$.

Similarly,

$$\frac{m}{\gamma_{k,m}}\otimes\gamma_{k,m}-\frac{m}{\gamma\prime}\otimes\gamma\prime$$

is in the span of $Y_{\leq m}$.

But

$$\frac{m}{\gamma_{j,m}}\otimes\gamma_{j,m}-\frac{m}{\gamma_{k,m}}\otimes\gamma_{k,m}$$

is in Y_m itself.

These three add up to

$$\frac{m}{\gamma}\otimes\gamma-\frac{m}{\gamma\prime}\otimes\gamma\prime$$

□

3. Second Generating Set

We now give the analogous results for the simplicial complexes $L_{<m}$.

For each $L_{<m}$ which is disconnected let the $C_{i,m}$'s be the connected components as in the previous case. Let $\gamma_{i,m}$ be any vertex in $C_{i,m}$. Define the set $Y_{m,L}$ by

$$Y_{m,L}=\left\{\frac{m}{\gamma_{i,m}}\otimes\gamma_{i,m}-\frac{m}{\gamma_{j,m}}\otimes\gamma_{j.m}\right\}\subset R\otimes_k R$$

using these new $\gamma_{i,m}$'s

Proposition 2. *For each m such that $L_{<m}$ is disconnected and for each pair $\gamma_{i,m}, \gamma_{j,m}$ where $i \neq j$,*

$$m = \operatorname{lcm}(\gamma_{i,m}, \gamma_{j,m})$$

.

Proof. This is even easier to prove than Proposition 1, since if

$$m \neq \operatorname{lcm}(\gamma_{i,m}, \gamma_{j,m})$$

then there is actually an edge of $L_{<m}$ between the two γ's. □

Define $Y_{\leq m,L}$ and Y_L analogously to $Y_{\leq m}$ and Y
T_m, and T are defined as before.

Theorem 2. *Y_L spans T.* [1]

Proof. Exactly as before we prove by induction on the *lcm-lattice* that

$$Y_{\leq m,L} \text{ spans } T_m$$

Let γ and $\gamma\prime$ be any element of $G(I)$. Let $m = n_{\gamma,\gamma\prime}$ and consider the element

$$\frac{m}{\gamma} \otimes \gamma - \frac{m}{\gamma\prime} \otimes \gamma\prime$$

It suffices to prove that this is in the span of $Y_{\leq m,L}$.

Case 1:

γ and $\gamma\prime$ are vertices of the same connected component of $L_{<m}$.
Then there exists a sequence of elements of $G(I)$,

$$\gamma = \gamma_1, \gamma_2, \dots, \gamma_l = \gamma\prime$$

such that the edge $\{\gamma_i, \gamma_{i+1}\}$ is in $L_{\leq m}$ for $i = 1, 2, \dots, l-1$.
Let $n_i = \operatorname{lcm}(\gamma_i, \gamma_{i+1})$. for $i = 1, 2, \dots, l-1$.
Then $n_i | m$, but $n_i \neq m$. Thus

$$\frac{n_i}{\gamma} \otimes \gamma_i - \frac{n_i}{\gamma_{i+1}} \otimes \gamma_{i+1}$$

is in T_{n_i}.
By induction we assume that

$$\frac{n_i}{\gamma} \otimes \gamma_i - \frac{n_i}{\gamma_{I+1}} \otimes \gamma_{i+1}$$

is in the span of $Y_{\leq n_i,L}$ and hence in the span of $Y_{\leq m,L}$.
The argument now goes exactly as before to complete Case 1.

Case 2:

γ and $\gamma\prime$ are vertices of different connected components of $L_{\leq m}$.

[1] The author needed the first generating set. He slightly modified the proof given here, which is by Victor Reiner, to get the proof in the previous section.

The argument again goes as before, and I leave it as an exercise to complete the proof of the theorem. □

4. Minimal Generating Sets for the First Syzygies

By Theorem 2.1, page 523 of [GPW], and Proposition 1.1 of [BH] we have that the minimal number of 1st syzygies of I (which is the same as the minimal number of 2nd syzygies of R/I) is given by the formula

$$b_2(R/I) = \sum_m \dim \tilde{H}_0(\Gamma_m) = \sum_m \dim \tilde{H}_0(L_{<m})$$

where m is quantified over all monomials in the *lcm-lattice*

For each m in the *lcm-lattice*, $\dim \tilde{H}_0(\Gamma_m) = \dim \tilde{H}_0(L_{<m})$ is one less than k_m, the number of connected components of Γ_m, respectively, $L_{<m}$.

For each m such that Γ_m, respectively, $L_{<m}$ is disconnected we consider the set

$$Z_m, \text{ respectively, } Z_{m,L} = \left\{ \frac{m}{\gamma_{i,m}} \otimes \gamma_{i,m} - \frac{m}{\gamma_{1,m}} \otimes \gamma_{1,m} \right\}$$

for $i = 2, 3, \ldots, k_m$.

The cardinality of this set is also one less than k_m. On the other hand, by taking differences, we see that this set spans Y_m, respectively,$Y_{m,L}$. Quantifying over all m in the *lcm-lattice*, we see that the set

$$Z, \text{ respectively, } Z_L = \bigcup_m Z_m, \text{ respectively, } Z_{m,L}$$

spans Y, respectively Y_L and hence T by Theorem 1, respectively Theorem 2.

By the formula cited above from [GPW] we see that the cardinality of Z, respectively, Z_L is the number of minimal generators of the first syzygies.

Since Z, respectively Z_L has the correct cardinality and spans T, each must be a minimal generating set for the first syzygies.

References

[BH] W. Bruns and J. Herzog, *Semigroup rings and simplicial complexes,* J. Pure. Appl. Algebra **122** (1997), 185-208.

[GPW] V. Gasharov, I. Peeva and V. Welker, *The lcm-lattice in monomial resolutions,* Math. Res. Lett. **6,** (1999), 521-532.